Clairaut's Thm :

If f_{xy} and f_{yx} both exists and continuous on domain Ω,
then $f_{xy} = f_{yx}$ on Ω.

$$
\begin{aligned}
f_{xxyyy} &= ((f_x)_{xy})_{yy} \\
&= ((f_x)_{yx})_{yy} \\
&= f_{xyxyy} \\
&= ((f_{xy})_{xy})_y \\
&= ((f_{yx})_{yx})_y \\
&= f_{yxyxy} \\
&= \cdots = f_{yyxxy} \\
&= f_{yyyxx} = \cdots
\end{aligned}
$$

For k-th partial derivatives,
We can change the order as we wish
by switching the 2 successive partial derivatives
provided that f is C^k

(C^k : all k-th order partial derivatives exist and continuous.)

§14.3

56. The fifth-order partial derivative $\partial^5 f/\partial x^2 \partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x or y? Try to answer without writing anything down.

a. $f(x, y) = y^2x^4e^x + 2$

If we treat everything unrelated to x as constant:

$$f(x,y) = C_1 x^4 e^x + C_2$$

If we treat everything unrelated to y as constant:

$$f(x,y) = C_1 y^2 + C_2$$

easier to differentiate.

$f_y = 2C_1 y$ $\quad f_{yy} = 2C_1$

$f_{yyy} = 0$ $\quad f_{yyyxx} = 0$.

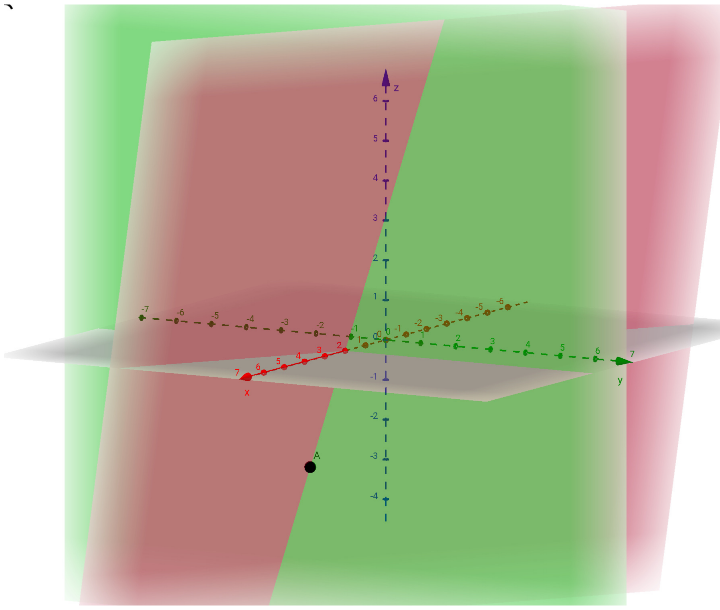
61. Let $f(x, y) = 2x + 3y - 4$. Find the slope of the line tangent to this surface at the point $(2, -1)$ and lying in the **a.** plane $x = 2$ **b.** plane $y = -1$.

$x = 2$,

$f_y(x, y) = 3$

$f_y(2, -1) = 3$

$\therefore$ the slope $= 3$



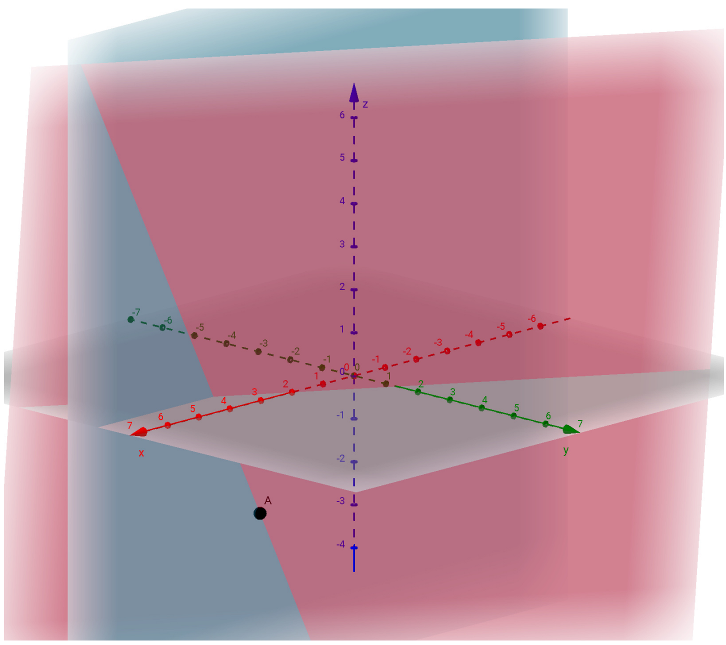
$z = f(x, y)$ $\quad$ $x = 2$

$y = -1$,

$f_x(x, y) = 2$

$f_x(2, -1) = 2$

$\therefore$ the slope $= 2$



$y = -1$ $\quad$ $z = f(x, y)$

65. Find the value of $\partial z/\partial x$ at the point (1, 1, 1) if the equation

$$xy + z^3x - 2yz = 0$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

$$xy + z^3x - 2yz = 0$$

$$\frac{\partial}{\partial x}\left(xy + z^3x - 2yz\right) = \frac{\partial}{\partial x}(0)$$

$$y + z^3 + 3z^2x\frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial x} = 0$$

$$(3z^2x - 2y)\frac{\partial z}{\partial x} = -y - z^3$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = -2$$

We don't differentiate y with respect to x because x, y are independent variables.

Or we can differentiate y w.r.t. x and then put $\frac{\partial y}{\partial x} = 0$

71. Let $f(x, y) = \begin{cases} y^3, & y \geq 0 \\ -y^2, & y < 0. \end{cases}$

Find f_x, f_y, f_{xy}, and f_{yx}, and state the domain for each partial derivative.

$f_x = 0 \qquad f_{xy} = 0$

For $y > 0, \quad f_y = 3y^2$

$y < 0, \quad f_y = -2y$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^3 - 0}{h} = 0$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h^2 - 0}{h} = 0$$

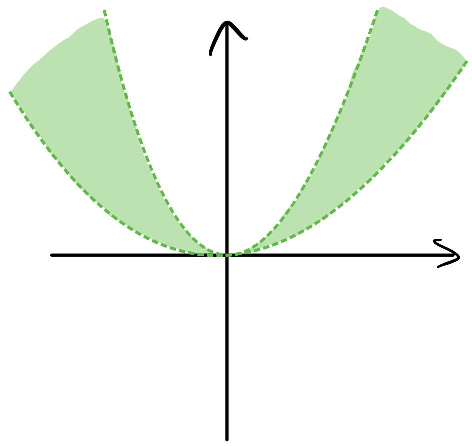
$\therefore f_y(x, 0) = 0$

$$\therefore f_y(x, y) = \begin{cases} 3y^2, & y \geq 0 \\ -2y, & y < 0 \end{cases}$$

$f_{yx} = 0$

92. Let $f(x, y) = \begin{cases} 0, & x^2 < y < 2x^2 \\ 1, & \text{otherwise.} \end{cases}$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$.



$$\lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{1 - 1}{h} = 0$$

$\therefore f_x(0, 0)$ exists and $= 0$

$$\lim_{h \to 0} \frac{f(0, 0+h) - f(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{1 - 1}{h} = 0$$

$\therefore f_y(0, 0)$ exists and $= 0$.

To show not differentiable :

Method 1 : By definition :

Def (differentiability).

$f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be differentiable at $\vec{a}$

if

(i). all 1st order partial derivatives exists at $\vec{a}$.

(ii).

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \varepsilon(\vec{x})$$

linear terms, can be rewritten as : $C_0 + C_1 x_1 + C_2 x_2 + \dots + C_n x_n$

Error terms

where $\frac{\varepsilon(\vec{x})}{\|\vec{x} - \vec{a}\|} \to 0$ as $\vec{x} \to \vec{a}$.

Recall : Taylor Series for $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = \underbrace{f(a) + f'(a)(x-a)}_{\text{linear terms}} + \underbrace{\frac{f''(a)}{2!}(x-a)^2 + \dots}_{\text{Error terms}}$$

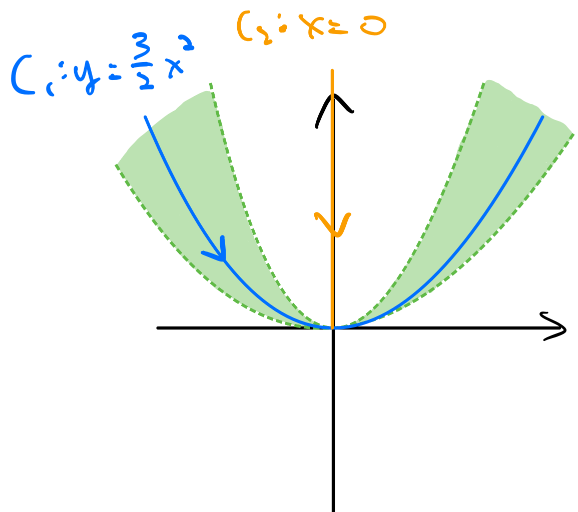
$$\lim_{x \to a} \frac{\frac{f''(a)}{2!}(x-a)^2 + \dots}{|x-a|} = 0 .$$

(iii) :

$$\varepsilon(x,y) = f(x,y) - \left[f(0,0) + \frac{\partial f}{\partial x}\bigg|_{(0,0)} (x-0) + \frac{\partial f}{\partial y}\bigg|_{(0,0)} (y-0) \right]$$

$$= f(x,y) - 1$$

$$\varepsilon(x,y) = \begin{cases} -1, & x^2 < y < 2x^2 \\ 0, & \text{else}. \end{cases}$$



$$\lim_{\substack{(x,y)\to(0,0) \\ \text{on } C_2}} \frac{\varepsilon(x,y)}{\|(x,y)-(0,0)\|} = \lim \frac{0}{|y|} = 0$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{on } C_1}} \frac{\varepsilon(x,y)}{\|(x,y)-(0,0)\|} = \lim \frac{-1}{\underline{\underline{\sqrt{x^2+y^2}}}}$$

$\leftarrow$ constant

$\to 0^+$ as $(x,y)\to(0,0)$

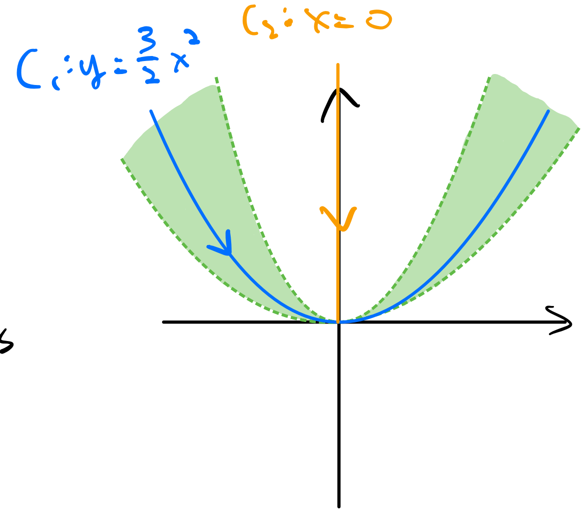
$\therefore$ $\varepsilon(x,y)$ has no limit at $(0,0)$.

$\therefore$ $f(x,y)$ is not differentiable at $(0,0)$.

Or Method 2:

differentiable implies continuity,

then not continuous implies not differentiable



On C_1, $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{on } C_1}} f(x,y) = 0$

On C_2, $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{on } C_2}} f(x,y) = 1$

$\therefore$ $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ not exist.

$\therefore$ f not continuous at $(0,0)$

$\therefore$ f not differentiable at $(0,0)$.